Differential Equations - Sample

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Differential Equations

Differential equations are mathematical expressions that describe how a quantity changes over time, based on its rate of change. They are used to model a wide range of phenomena in physics, engineering, biology, and other fields. Second-order differential equations, in particular, describe how the rate of change of a quantity is itself changing over time.

In this chapter, we will focus on methods for solving second-order ordinary differential equations (ODEs). These equations involve functions and their first and second derivatives, and are commonly used to model a variety of natural phenomena. We will cover two main methods for solving second-order ODEs: the integrating factor method and the auxiliary equation method. These methods involve transforming the original differential equation into a simpler form, which can then be solved using basic calculus techniques. By the end of this chapter, you will be able to apply these methods to solve a variety of second-order ODEs.

Integrating Factors

Integrating factors are used to solve certain types of ordinary differential equations (ODEs) that cannot be solved by direct integration. An integrating factor is a function that is multiplied by both sides of an ODE to transform it into an exact differential equation, which can then be solved by integrating both sides.
Integrating factors are particularly useful when dealing with ODEs that are not exact or that cannot be made exact by a simple manipulation. In such cases, the integrating factor can be used to effectively convert the ODE into an exact form, simplifying the process of finding its solution.
To solve an ODE:

1. Write the ODE in standard form:

$$\frac{dy}{dx} + P(x)y = Q(x)$$

2. Identify the integrating factor, denoted by $I(x)$, as a function that satisfies the following condition:

$$I(x)e^{\int P(x)dx} = \text{constant}$$

3. Multiply both sides of the ODE by the integrating factor $I(x)$:

$$I(x)\frac{dy}{dx} + I(x)P(x)y = I(x)Q(x)$$

4. Rewrite the left-hand side of the equation as the derivative of a product:

$$\frac{d}{dx}\left(I(x)y\right) = I(x)Q(x)$$

5. Integrate both sides with respect to x:

$$I(x)y = \int I(x)Q(x)dx + C$$

where C is the constant of integration.

6. Solve for y:

$$y = \frac{1}{I(x)}\left(\int I(x)Q(x)dx + C\right)$$

These steps can be used to find the general solution to the ODE. The integrating factor is a key element of this method because it transforms the original equation into a form that can be easily integrated. The integrating factor is chosen such that when multiplied by both sides of the ODE, the left-hand side can be written as the derivative of a product. This allows us to integrate both sides of the equation and find a solution for y. The constant of integration C can be determined from initial or boundary conditions.

Second order non-homogeneous ODEs

To solve a second-order non-homogeneous ODE using an integrating factor, we can first rewrite the equation in a form that allows us to apply the method. Suppose we have a second-order ODE of the form:

$$y'' + p(x)y' + q(x)y = f(x)$$

To use the integrating factor method, we first multiply both sides of the equation by an integrating factor $u(x)$, chosen so that the left-hand side of the equation becomes the derivative of a product:

$$u(x)y'' + u(x)p(x)y' + u(x)q(x)y = u(x)f(x)$$

Now, we want the left-hand side of the equation to be the derivative of a product. This suggests that we should choose $u(x)$ so that:

$$u(x)p(x) = \frac{d}{dx}u(x)$$

which means that $u(x)$ is an exponential function of the integral of $p(x)$:

$$u(x) = e^{\int p(x)dx}$$

Multiplying both sides of the equation by this integrating factor, we get:

$$e^{\int p(x)dx}y'' + e^{\int p(x)dx}p(x)y' + e^{\int p(x)dx}q(x)y = e^{\int p(x)dx}f(x)$$

The left-hand side can now be simplified by using the product rule, giving:

$$\frac{d}{dx}\left(e^{\int p(x)dx}y'\right) + e^{\int p(x)dx}q(x)y = e^{\int p(x)dx}f(x)$$

Integrating both sides with respect to x and simplifying, we get:

$$y(x) = \frac{1}{u(x)}\left(\int u(x)f(x)dx + C_1\right) + C_2$$

where C_1 and C_2 are constants of integration.
Here's an example:
Suppose we want to solve the second-order ODE:

$$y'' - 2y' + y = x^2$$

We can rewrite this equation in the form:

$$y'' - 2y' + y = f(x)$$

where $f(x) = x^2$. The integrating factor is then:

$$u(x) = e^{\int -2dx} = e^{-2x}$$

Multiplying both sides of the equation by $u(x)$, we get:

$$e^{-2x}y'' - 2e^{-2x}y' + e^{-2x}y = e^{-2x}x^2$$

The left-hand side of this equation is the derivative of $e^{-2x}y'$, so we can simplify further:

$$\frac{d}{dx}\left(e^{-2x}y'\right) + e^{-2x}y = e^{-2x}x^2$$

Integrating both sides with respect to x, we get:

$$e^{-2x}y' + \frac{1}{2}e^{-2x}y = -\frac{1}{2}e^{-2x}x^2 + C_1$$

where C_1 is a constant of integration. Now, we can use the product rule in reverse to write this equation as:

$$\frac{d}{dx}\left(e^{-2x}y\right) = -\frac{1}{2}e^{-2x}x^2 + C_1$$

Integrating both sides again with respect to x, we get:

$$e^{-2x}y = \int -\frac{1}{2}e^{-2x}x^2 dx + C_1x + C_2$$

where C_2 is another constant of integration. Evaluating the integral on the right-hand side, we get:

$$e^{-2x}y = -\frac{1}{4}x^2e^{-2x} - \frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x} + C_1x + C_2$$

Simplifying, we get:

$$y(x) = \frac{1}{e^{-2x}}\left(-\frac{1}{4}x^2 - \frac{1}{2}x - \frac{1}{4}e^{2x} + C_1xe^{2x} + C_2e^{2x}\right)$$

which can be further simplified as:

$$y(x) = \frac{1}{4}\left(x^2 + 2x + 1\right) + C_1xe^{2x} + C_2e^{2x}$$

So the general solution to the given second-order ODE is:

$$y(x) = \frac{1}{4}\left(x^2 + 2x + 1\right) + C_1xe^{2x} + C_2e^{2x}$$

where C_1 and C_2 are constants of integration.

Second order homogeneous ODEs

The integrating factor method is a powerful technique for solving first-order linear differential equations. However, when it comes to solving homogeneous second-order linear differential equations, the integrating factor method is not applicable. We need to resort to a different approach to solve these types of differential equations.

One of the methods used to solve homogeneous second-order linear differential equations is the auxiliary equation method. The basic idea of this method is to assume a solution in the form of a function with exponential behavior and use it to derive an algebraic equation, known as the characteristic equation.

The characteristic equation is derived by substituting the assumed solution into the homogeneous differential equation and solving for the constants. The constants that arise in the solution correspond to the roots of the characteristic equation. These roots will help us to construct the general solution of the differential equation.

A homogeneous second-order ordinary differential equation (ODE) is an equation of the form:

$$ay'' + by' + cy = 0$$

where a, b, and c are constants.
To solve a homogeneous second-order ODE, we can first find the auxiliary equation by assuming that the solution has the form $y = e^{rx}$, where r is a constant. Substituting this form into the ODE, we get:

$$a(r^2 e^{rx}) + b(re^{rx}) + c(e^{rx}) = 0$$

Dividing both sides by e^{rx}, we get:

$$ar^2 + br + c = 0$$

This equation is called the auxiliary equation. We can solve for the roots of the auxiliary equation using the quadratic formula:

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

There are three possible cases depending on the nature of the roots of the auxiliary equation:

1. Real and distinct roots: If the roots of the auxiliary equation are real and distinct, then the general solution to the homogeneous second-order ODE is given by:

   $$y = C_1 e^{r_1 x} + C_2 e^{r_2 x}$$

   where C_1 and C_2 are constants determined by the initial conditions.

2. Real and repeated roots: If the roots of the auxiliary equation are real and repeated, then the general solution to the homogeneous second-order ODE is given by:

   $$y = C_1 e^{rx} + C_2 x e^{rx}$$

   where C_1 and C_2 are constants determined by the initial conditions.

3. Complex roots: If the roots of the auxiliary equation are complex, say $r = \alpha + i\beta$ and $r = \alpha - i\beta$, where α and β are real, then the general solution to the homogeneous second-order ODE is given by:

   $$y = e^{\alpha x}(C_1 \cos \beta x + C_2 \sin \beta x)$$

   where C_1 and C_2 are constants determined by the initial conditions.

Note that in the complex roots case, the real part of the solution decays or grows exponentially depending on the sign of α, while the imaginary part oscillates sinusoidally with frequency β.

Trial Functions

Trial functions are functions that are used to guess a particular integral for a given differential equation. They are typically chosen based on the form of the nonhomogeneous term in the differential equation. For example, if the nonhomogeneous term is a polynomial, a trial function might be a polynomial of the same degree, or a polynomial multiplied by an exponential function or a trigonometric function.

The idea behind using a trial function is to guess the form of the particular integral, substitute it into the differential equation, and then solve for the unknown coefficients in the particular integral that make it satisfy the equation. This approach is often useful in finding particular integrals for linear differential equations with constant coefficients.

The process of finding a particular integral using trial functions involves several steps, including:

1. Identifying the form of the nonhomogeneous term in the differential equation.
2. Guessing a trial function based on the form of the nonhomogeneous term.
3. Computing the derivatives of the trial function and substituting them into the differential equation.
4. Solving for the unknown coefficients in the trial function that make it satisfy the differential equation.
5. Combining the particular integral with the general solution of the homogeneous equation to obtain the general solution of the nonhomogeneous equation.

The choice of trial functions depends on the form of the non-homogeneous term in the differential equation. Here are some commonly used trial functions:

Form	Trial Function
Constant	$y_p = A$
Polynomial	$y_p = C_1x^n + C_2x^{n-1} + \cdots + C_m$
Exponential	$y_p = Ce^{rx}$
Trigonometric	$y_p = C_1 \cos rx + C_2 \sin rx$
Power Series	$y_p = \sum_{n=0}^{\infty} c_n x^n$
Product of polynomial and exponential	$y_p = x^n e^{rx}(a_0 + a_1x + \cdots + a_mx^m)$
Product of polynomial and trigonometric	$y_p = x^m(a_0 + a_1x + \cdots + a_nx^n)(A \cos rx + B \sin rx)$

Note that these are not the only trial functions that can be used, but they are among the most commonly used ones. The choice of trial function depends on the specific form of the non-homogeneous term, and sometimes multiple trial functions may need to be tried before finding the correct one.

Transforming Differential Equations with Substitutions

Using substitutions to transform differential equations is a common technique used to simplify the process of solving differential equations. In some cases, a clever substitution can transform a complicated differential equation into a simpler one that is easier to solve.

The $v = \frac{y}{x}$ substitution

One common type of substitution is the substitution $v = \frac{y}{x}$, which is particularly useful for solving equations of the form $\frac{dy}{dx} = f(\frac{y}{x})$. This substitution allows us to rewrite the differential equation in terms of a single variable v, which can often be easier to work with.

To make this substitution, we first write $\frac{dy}{dx}$ in terms of v using the chain rule:

$$\frac{dy}{dx} = \frac{dv}{dx} \cdot x + v \cdot \frac{d}{dx}(x)$$

Simplifying, we get:

$$\frac{dy}{dx} = x\frac{dv}{dx} + v$$

We can then substitute $v = \frac{y}{x}$, which gives us:

$$\frac{dy}{dx} = x\frac{dv}{dx} + \frac{y}{x}$$

This equation is now in terms of v and x, which we can often solve more easily than the original equation in terms of y and x.

For example, consider the differential equation $\frac{dy}{dx} = x^2 + y^2$. Using the substitution $v = \frac{y}{x}$, we can rewrite this equation as:

$$\frac{dy}{dx} = x^2 + y^2 = x^2\left(1 + v^2\right)$$

Dividing both sides by x^2 and simplifying, we get:

$$\frac{dv}{dx} + v^2 = 1$$

This equation is now separable and can be solved using standard techniques. Once we have solved for v, we can then substitute back to find y in terms of x.

The $x = e^t$ substitution

Another useful substitution is $x = e^t$. This substitution is useful for equations of the type:

$$ax^2\frac{d^2y}{dx^2} + bx\frac{dy}{dx} + cy = f(x)$$

Substituting $x = e^t$ gives:

$$a\frac{d^2y}{dt^2} + (b - a)\frac{dy}{dt} + cy = f(e^t)$$

This is a linear second-order differential equation in y with constant coefficients, which can be solved using standard methods such as the characteristic equation or Laplace transforms.

Let's work through an example to see how this substitution works in practice.

For example, solve the differential equation $2x^2\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + y = x^3$ using the substitution $x = e^t$.

We start by substituting $x = e^t$:

$$2\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + y = e^{3t}$$

Next, we solve the homogeneous equation:

$$2\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + y = 0$$

The characteristic equation is $2r^2 - 2r + 1 = 0$, which has roots $r = 1/2 \pm i\sqrt{3}/2$. Therefore, the general solution to the homogeneous equation is:

$$y_h = c_1 e^{t/2}\cos\left(\frac{\sqrt{3}}{2}t\right) + c_2 e^{t/2}\sin\left(\frac{\sqrt{3}}{2}t\right)$$

Next, we find a particular solution to the non-homogeneous equation. We can use the method of undetermined coefficients and guess that the particular solution has the form $y_p = Ax^3$. Substituting $x = e^t$ gives $y_p = Ae^{3t}$. We can substitute this into the original equation to find A:

$$2e^{6t} - 8e^{4t} + e^{3t} = Ae^{3t}$$

Solving for A gives $A = 1/2$, so the particular solution is $y_p = \frac{1}{2}e^{3t}$.

Therefore, the general solution to the non-homogeneous equation is:

$$y = c_1 e^{t/2} \cos\left(\frac{\sqrt{3}}{2}t\right) + c_2 e^{t/2} \sin\left(\frac{\sqrt{3}}{2}t\right) + \frac{1}{2}e^{3t}$$

The $z = \frac{dy}{dx}$ substitution

When x and y do not both appear explicitly

In some differential equations, the variables x and y do not both appear explicitly. In such cases, we can use a substitution of the form $z = \frac{dy}{dx}$ to convert the equation into one that is in terms of x and z. Let's take an example to see how this works.

Consider the differential equation:

$$y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = x$$

To use the substitution $z = \frac{dy}{dx}$, we first need to find $\frac{d^2y}{dx^2}$ in terms of x and z. We can differentiate both sides of $z = \frac{dy}{dx}$ with respect to x to get:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{dz}{dx}$$

Now, we can substitute $\frac{dz}{dx}$ and z into the original differential equation to get:

$$y\frac{dz}{dx} + z^2 = x$$

This is now an equation in terms of x and z, and we can solve it using methods for first-order differential equations.

Another example is the differential equation: $\frac{dy}{dx} = \frac{1}{x+y}$. Here, we can use the substitution $z = \frac{dy}{dx}$ to get:

$$z = \frac{1}{x+y}$$

Differentiating both sides with respect to x gives:

$$\frac{dz}{dx} = -\frac{1}{(x+y)^2} \cdot \frac{dy}{dx} = -\frac{z^2}{(x+y)}$$

Now we can substitute z and $\frac{dz}{dx}$ into the original differential equation to get:

$$z\frac{dz}{dx} = -\frac{z^2}{x} - z$$

This is now an equation in terms of x and z, and we can solve it using methods for first-order differential equations.

The substitution $z = \frac{dy}{dx}$ is useful when the original differential equation involves $\frac{dy}{dx}$ but not y explicitly. By using this substitution, we can transform the equation into one that is in terms of x and z, and solve it using methods for first-order differential equations.

When the equation does not contain x explicitly

The substitution $z = \frac{dy}{dx}$ can also be used to solve differential equations where x does not appear explicitly. In this case, the equation is transformed into an equation in terms of z and y.

For example, consider the differential equation:

$$y\frac{d^2y}{dx^2} - \left(\frac{dy}{dx}\right)^2 = 0$$

Here, x does not appear explicitly. Letting $z = \frac{dy}{dx}$, we have:

$$y\frac{dz}{dx} - z^2 = 0$$

This is a separable first-order differential equation that can be solved using separation of variables:

$$\frac{1}{z^2}\frac{dz}{y} = \frac{1}{x}dx$$

Integrating both sides, we get:

$$-\frac{1}{z} = \ln|x| + C$$

where C is the constant of integration. Substituting back $z = \frac{dy}{dx}$, we get:

$$\frac{dy}{dx} = -\frac{1}{\ln|x| + C}$$

This is a first-order separable differential equation that can be solved using separation of variables:

$$\frac{dy}{\ln|x| + C} = -dx$$

Integrating both sides, we get:

$$\ln|\ln|x| + C| = -x + D$$

where D is another constant of integration. Solving for y, we get:

$$y(x) = \pm\sqrt{e^{D-(\ln|\ln|x|+C)}}$$

which simplifies to:

$$y(x) = \pm\sqrt{\frac{e^D}{|\ln|x| + C|}}$$

Therefore, the general solution to the differential equation is:

$$y(x) = \pm\sqrt{\frac{A}{|\ln|x| + B|}}$$

where $A = e^D$ and $B = -C$.